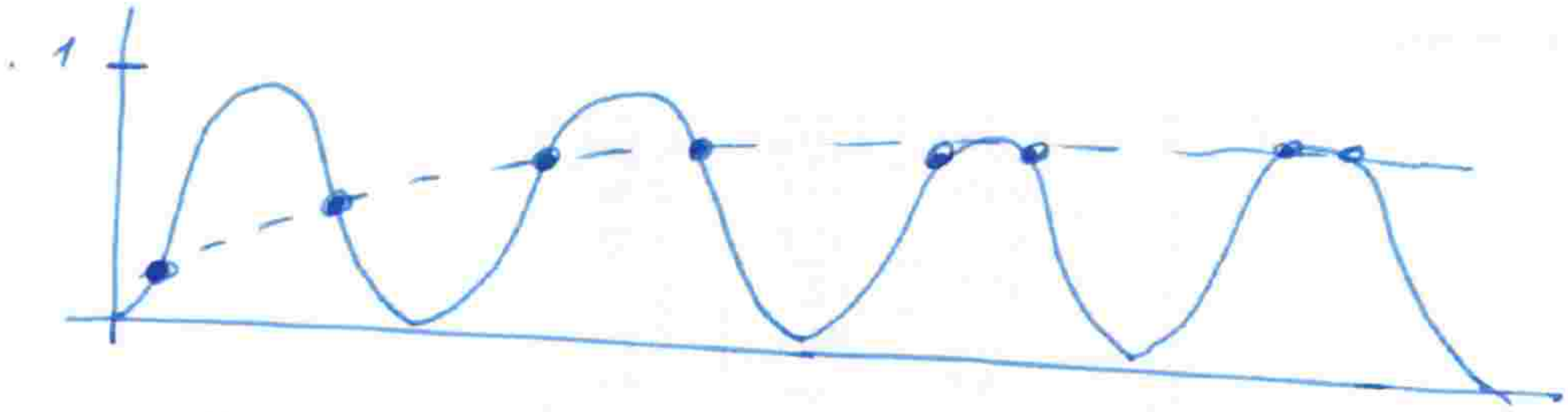
• Určete limitu posloupnosti $\sin^2(\pi\sqrt{n^2+n})$ ⌀



Platí, že pro $n \to \infty$ $\sqrt{n^2+n} \to n+\frac{1}{2}$ (jakási "asymptotická" posloupnost). Protože:

$$\lim_{n\to+\infty} \sqrt{n^2+n}-n = \lim \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim \frac{n}{\sqrt{n^2+n}+n} =$$

$$= \lim \frac{n}{n}\cdot\frac{1}{\sqrt{1+1/n}+1} = \lim_{n\to+\infty} \frac{1}{\sqrt{1+\underbrace{1/n}_{\to 0}}+1} = \frac{1}{2}$$

$\sin(x)$ je spojitá, Fce $\pi\sqrt{x^2+x}$ je prostá, tj: lze užít větu o limitě složené funkce (v jistém slova smyslu) a tedy

$$\lim_{n\to+\infty} \sin^2\left(\pi\sqrt{n^2+n}\right) = \lim_{n\to\infty} \sin^2\underbrace{\left(\pi\cdot n+\frac{\pi}{2}\right)}_{\sin(\pi n+\frac{\pi}{2}) = \pm 1} = 1$$

• Obdélník daného obsahu S – jaký může mít nejmenší obvod? ①

$$b = S/a$$

$$S = a \cdot b \Rightarrow b = \frac{S}{a}$$

$$O(a) = 2(a+b) = 2\left(a + \frac{S}{a}\right) = 2 \cdot \frac{a^2+S}{a}$$

$$O'(a) = 2\,\frac{2 \cdot a \cdot a - (a^2+S)}{a^2} = 2 \cdot \frac{a^2-S}{a^2} = 0$$

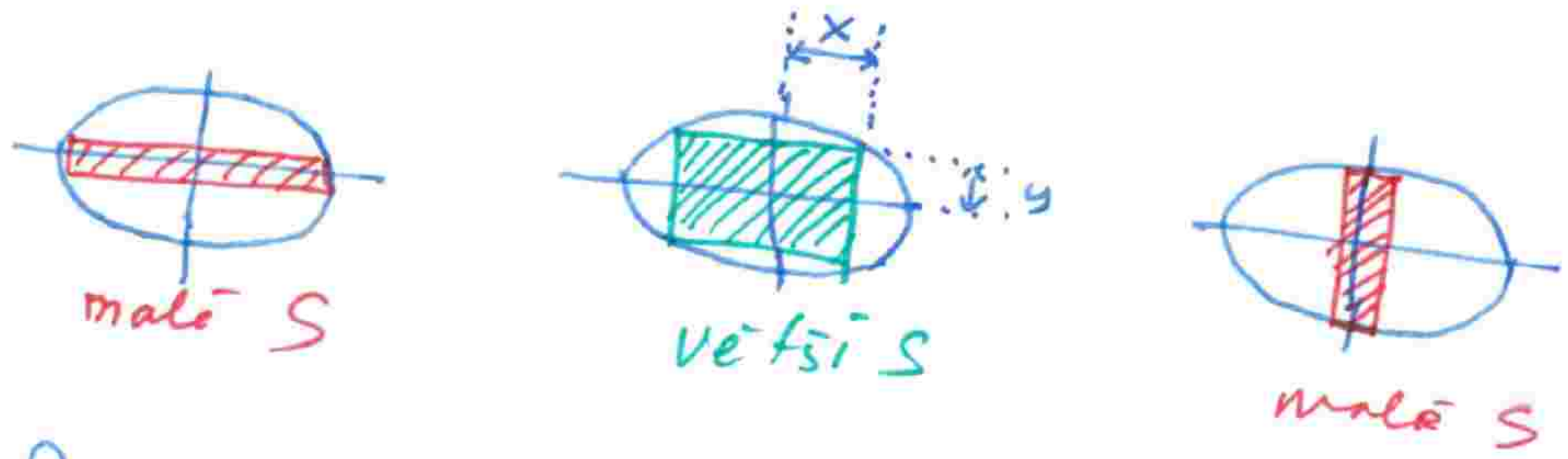
$$\Longleftrightarrow a^2 = S \Rightarrow a = \pm\sqrt{S}$$

$-\sqrt{S}$ nedává v úloze smysl, proto se na něj vykašlem a zbyde

$$a = \sqrt{S} \qquad b = \frac{S}{a} = \frac{S}{\sqrt{S}} = \sqrt{S} = a$$

Je to čtverec.

• Do elipsy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ vepište obdélník, jehož strany jsou ∥ s osami a který má maximální možný obsah.



$$S = 4 \cdot x \cdot y$$, zároveň $$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \Rightarrow y^2 = -a^2\left(\frac{x^2}{b^2} - 1\right)$$

omylem prohozeno

$$y = a\sqrt{1 - \frac{x^2}{b^2}}$$

$$S(x) = 4ax\sqrt{1 - \frac{x^2}{b^2}}$$ ← Derivace součinu a derivace složené funkce

$$S'(x) = 4a\sqrt{-\|-} + 4ax \cdot \left(-\frac{2x}{b^2}\right)\frac{1}{2\sqrt{-\|-}} = 0$$

$$\left\{ (\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \right\}$$

$$(f \cdot g)' = f'g + fg'$$
$$(f(g))' = g' f'(g)$$

$$S'(x) = 4a\left[\frac{2\left(1-\frac{x^2}{b^2}\right)-\frac{2x^2}{b^2}}{2\sqrt{1-\frac{x^2}{b^2}}}\right] = \underbrace{\frac{2a}{\sqrt{1-\frac{x^2}{b^2}}}}_{\text{vždy} > 0}\cdot\left(2-\frac{4x^2}{b^2}\right)=0 \quad (2)$$

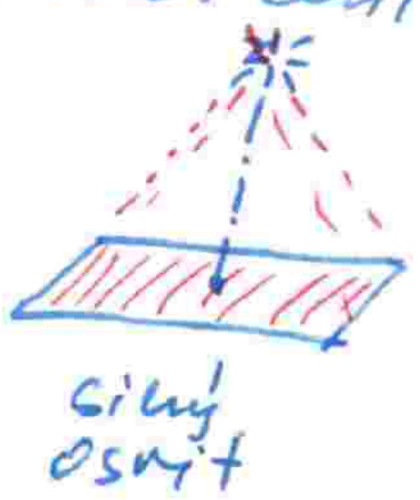
$$\Longleftrightarrow \quad 2-\frac{4x^2}{b^2}=0 \implies \frac{2x^2}{b^2}=1 \implies x=\frac{b}{\sqrt{2}}$$

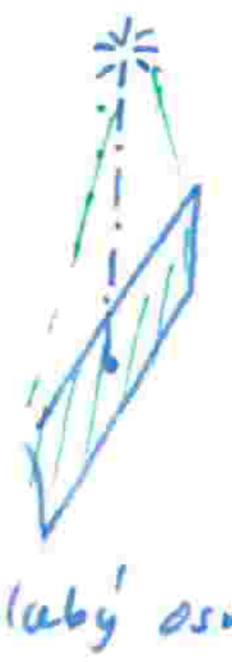
$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \implies \frac{b^2}{(\sqrt{2})^2 b^2}+\frac{y^2}{a^2}=1 \implies \frac{y^2}{a^2}=\frac{1}{2} \implies y=\frac{a}{\sqrt{2}}$$

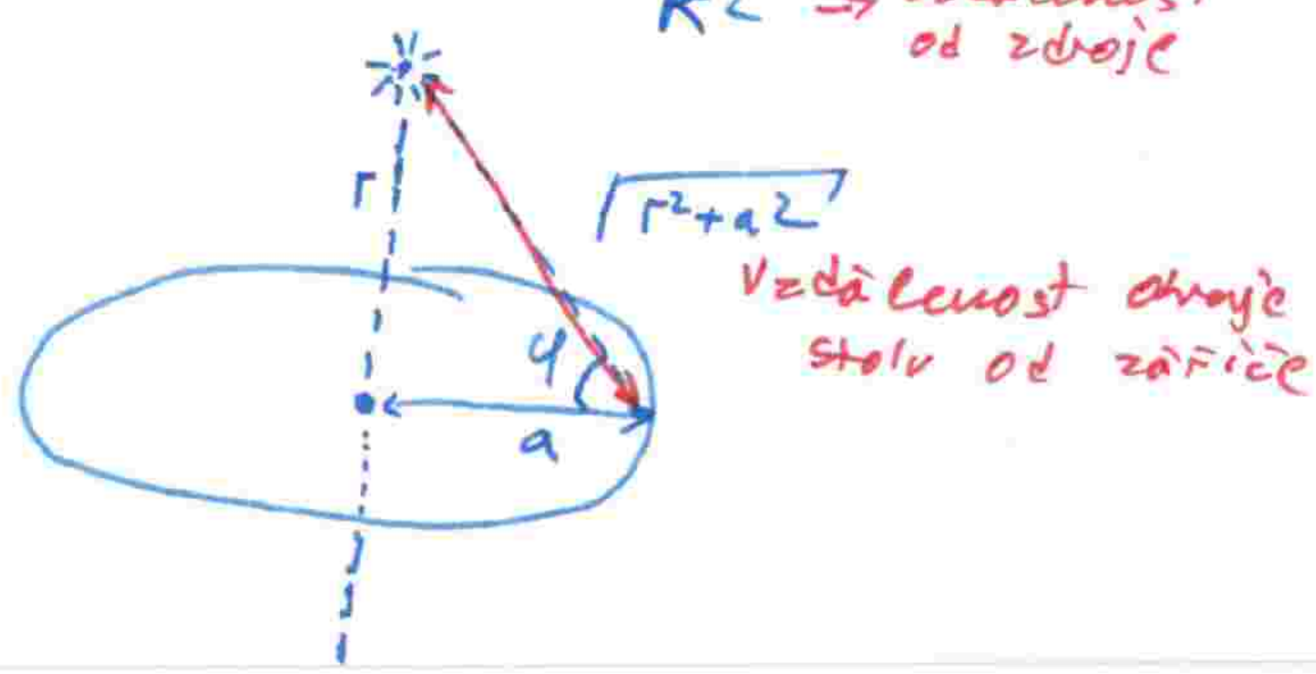
$$\boxed{x=\frac{b}{\sqrt{2}} \qquad y=\frac{a}{\sqrt{2}}}$$

• V jaké výšce nad středem kruhového stolu o poloměru a je třeba umístit svítidlo, aby osvětlení bylo maximálně jasné? Jas osvětlení je dán vzorcem

$$I = k\,\frac{\sin\varphi}{R^2}$$ → vzdálenost od zdroje







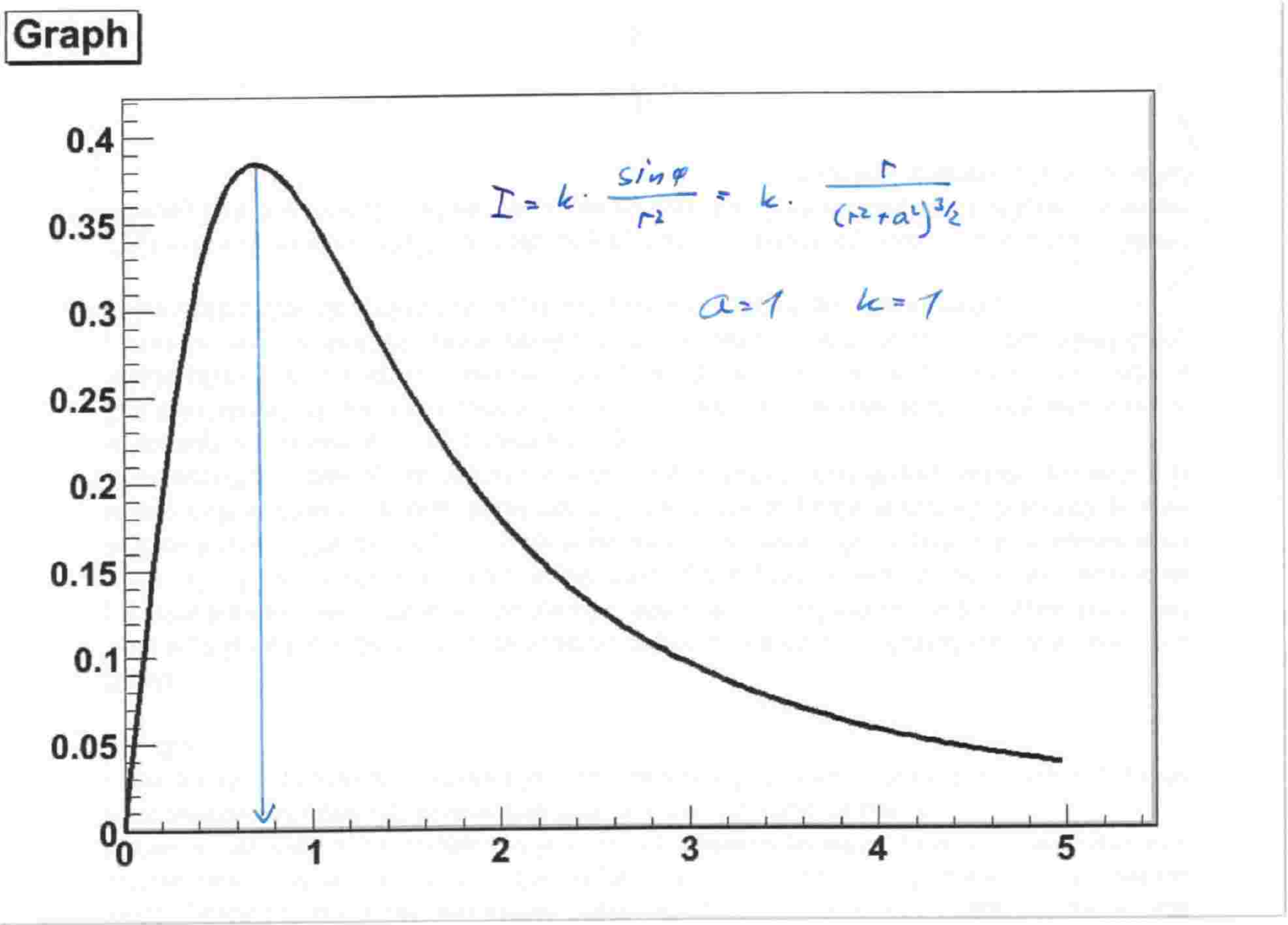
Neznámá je r; $a = \text{konst}$; φ dopočítáme.

$$\sin\varphi = \frac{r}{\sqrt{r^2+a^2}} \implies I(r) = k\cdot\frac{\frac{r}{\sqrt{r^2+a^2}}}{\left(\sqrt{r^2+a^2}\right)^2} = \frac{k\,r}{\sqrt{r^2+a^2}\,(r^2+a^2)} \implies$$

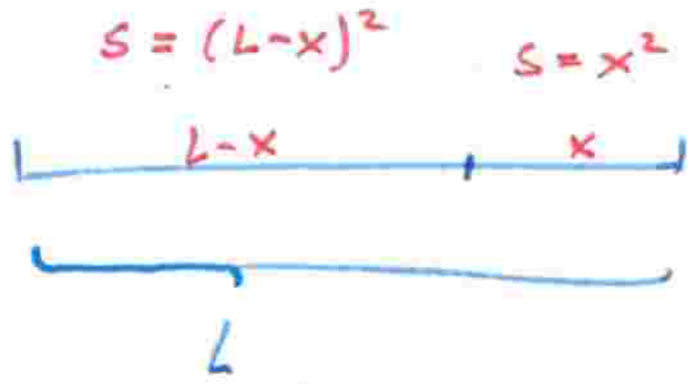
$$\implies I(r) = \frac{r}{(r^2+a^2)^{3/2}} \qquad I'(r) = \frac{(r^2+a^2)^{3/2} - r\cdot 2r\cdot\frac{3}{2}(r^2+a^2)^{1/2}}{(r^2+a^2)^3} =$$

$$= \frac{(r^2+a^2)^{1/2}}{(r^2+a^2)^3}\cdot\left[r^2+a^2-3r^2\right] = \frac{a^2-2r^2}{\underbrace{(r^2+a^2)^{5/2}}_{\text{vždy} > 0}} = 0 \iff a^2 = 2r^2$$

$$\implies r = \frac{a}{\sqrt{2}}$$


Graph
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
$I = k \cdot \frac{\sin\varphi}{r^2} = k \cdot \frac{r}{(r^2+a^2)^{3/2}}$
$a = 1 \quad k = 1$

• Rozdělte úsečku na dvě části tak, aby součet čtverců (obsahů) sestrojených nad oběma částmi byl minimální

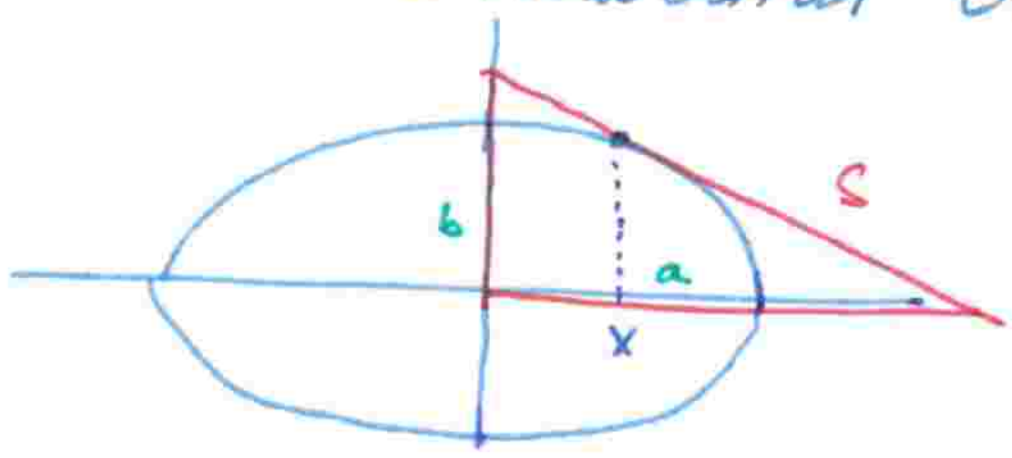


$$f(x) = (L-x)^2 + x^2$$

$$f'(x) = -2(L-x) + 2x = -2L + 4x = 0$$

$$4x = 2L \Rightarrow x = \frac{L}{2}$$

• Určete bod elipsy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tak, aby tečna k elipse sestrojená v tomto bodě vytvořila s osami souřadnic trojúhelník s minimálním obsahem



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$f(x) = y = b \cdot \sqrt{1 - \frac{x^2}{a^2}}$$

tečna : $k = f'(x) = b\left(-\frac{2x}{a^2}\right) \cdot \frac{1}{2\sqrt{1 - x^2/a^2}} = - \frac{bx}{a^2\sqrt{1 - x^2/a^2}}$

směrnice

$$\chi = k\xi + q \quad , \quad \text{bod } \left[x, b\sqrt{1 - x^2/a^2}\right]$$

$$b \cdot \sqrt{1 - x^2/a^2} = - \frac{bx \cdot x}{a^2\sqrt{1 - x^2/a^2}} + q$$

$$\Rightarrow q = b\sqrt{1 - x^2/a^2} + \frac{bx^2}{a^2\sqrt{1 - x^2/a^2}} = \frac{b \cdot a^2\left(1 - \frac{x^2}{a^2}\right) + bx^2}{a^2\sqrt{1 - x^2/a^2}} =$$

$$= \frac{b \cdot a^2 - bx^2 + bx^2}{a^2\sqrt{1 - x^2/a^2}} = \frac{b}{\sqrt{1 - \frac{x^2}{a^2}}}$$

pro $x \to 0$ je $q \to b$; pro $x \to a$ je $q \to +\infty$, což sedí.

Tj. rovnice přímky tečny v závislosti na x je

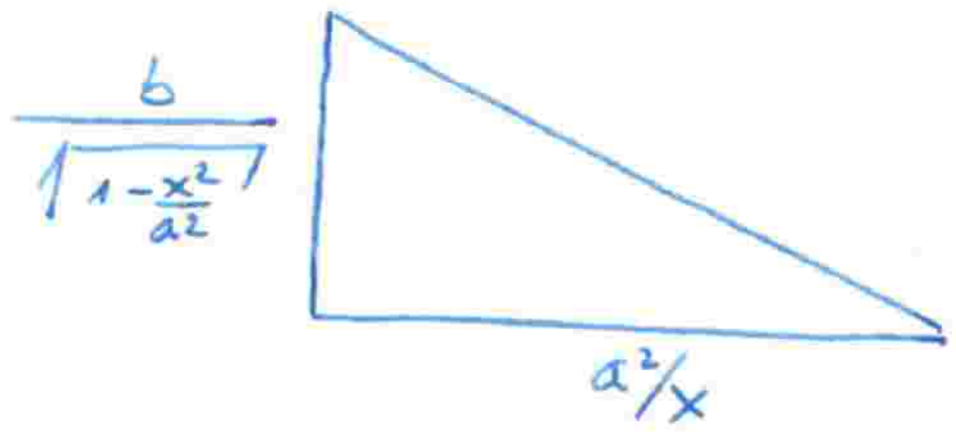
$$\chi = -\frac{bx}{a^2\sqrt{1-x^2/a^2}}\cdot\xi + \frac{b}{\sqrt{1-\frac{x^2}{a^2}}}$$

Abychom spočetli S trojúhelníka, zajímají nás průsečíky s osami. Pro průsečík s osou y už máme $\chi = \frac{b}{\sqrt{1-x^2/a^2}}$. Ten druhý

$$0 = -\frac{bx}{a^2\sqrt{1-x^2/a^2}}\xi + \frac{b}{\sqrt{1-x^2/a^2}}$$

$$\xi\cdot\frac{\cancel{b}x}{a^2\cancel{\sqrt{-\shortparallel-}}} = \frac{\cancel{b}}{\cancel{\sqrt{-\shortparallel-}}} \Rightarrow \xi = \frac{a^2}{x}$$

To také sedí, pro $x\to 0$ je $\xi\to\infty$ a pro $x\to a$ je $\xi\to a$



$$\Rightarrow 2S(x) = \frac{a^2\cdot b}{x\cdot\sqrt{1-\frac{x^2}{a^2}}}$$

$$S'(x) = \frac{0 - a^2 b\cdot\left[\sqrt{-\shortparallel-} + x\cdot\left(-\frac{2x}{a^2}\right)\cdot\frac{1}{2\sqrt{-\shortparallel-}}\right]}{x^2\left(1-x^2/a^2\right)} =$$

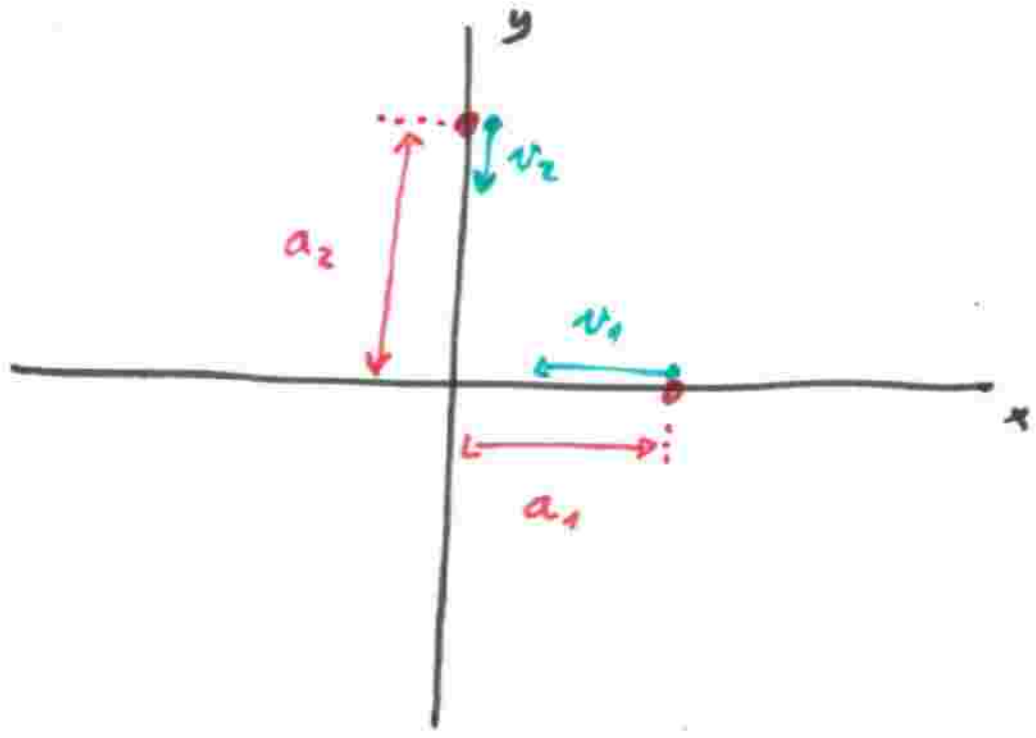
$$= \frac{-a^2 b}{x^2(1-x^2/a^2)}\left[\sqrt{-\shortparallel-} - \frac{x^2}{a^2}\frac{1}{\sqrt{-\shortparallel-}}\right] = \frac{\cdot}{\cdot}\cdot\frac{a^2\left(1-x^2/a^2\right) - x^2}{a^2\sqrt{-\shortparallel-}} =$$

$$= -\frac{b}{x^4(1-x^2/a^2)^{3/2}}\cdot\left[a^2 - x^2 - x^2\right] = \frac{b}{x^4(1-x^2/a^2)^{3/2}}\cdot\left[a^2 - 2x^2\right] = 0$$

$$\boxed{x = \frac{a}{\sqrt{2}}}\;; \quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow \frac{a^2}{2a^2}+\frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = \frac{1}{2} \Rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

$$\boxed{\xi = \frac{a^2}{x} = \frac{a^2}{a/\sqrt{2}} = a\cdot\sqrt{2}} \qquad \boxed{\chi = \frac{b}{\sqrt{1-\frac{a^2 2}{a^2}}} = \frac{b}{\sqrt{1/2}} = b\cdot\sqrt{2}}$$

• V rovině se po každé ose souřadnic pohybuje bod rychlostmi v_1 a v_2. V okamžiku $t=0$ jsou od počátku vzdáleny o a_1 a a_2. Ve kterém okamžiku budou mít minimální vzdálenost?



$$L = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} =$$
$$= \sqrt{(0-x_1)^2 + (y_2-0)^2} =$$
$$= \sqrt{x_1^2 + y_2^2}$$

$$x_1(t) = a_1 - v_1 \cdot t$$
$$y_2(t) = a_2 - v_2 \cdot t$$

$$\Rightarrow L(t) = \sqrt{(a_1 - v_1 t)^2 + (a_2 - v_2 t)^2}$$

$$L'(t) = \frac{-2v_1(a_1 - v_1 t) - 2v_2(a_2 - v_2 t)}{2\sqrt{-\|-}} =$$

$$= \frac{2(v_1^2 + v_2^2)t - 2(v_1 a_1 + v_2 a_2)}{\underbrace{2 -\|-}_{\text{vždy} > 0}} = 0$$

$$t = \frac{v_1 a_1 + v_2 a_2}{v_1^2 + v_2^2}$$

Sedí jednotky?

$$\left[\frac{v_1 a_1 + v_2 a_2}{v_1^2 + v_2^2}\right] = \frac{m s^{-1} \cdot m}{m^2 s^{-2}} =$$
$$= \frac{m^2 s^{-1}}{m^2 s^{-2}} = \frac{1}{s^{-1}} = s \checkmark$$

• $F(x) = x^4 - 8x^3 + 18x^2 - 11$

$D_F = \mathbb{R}$

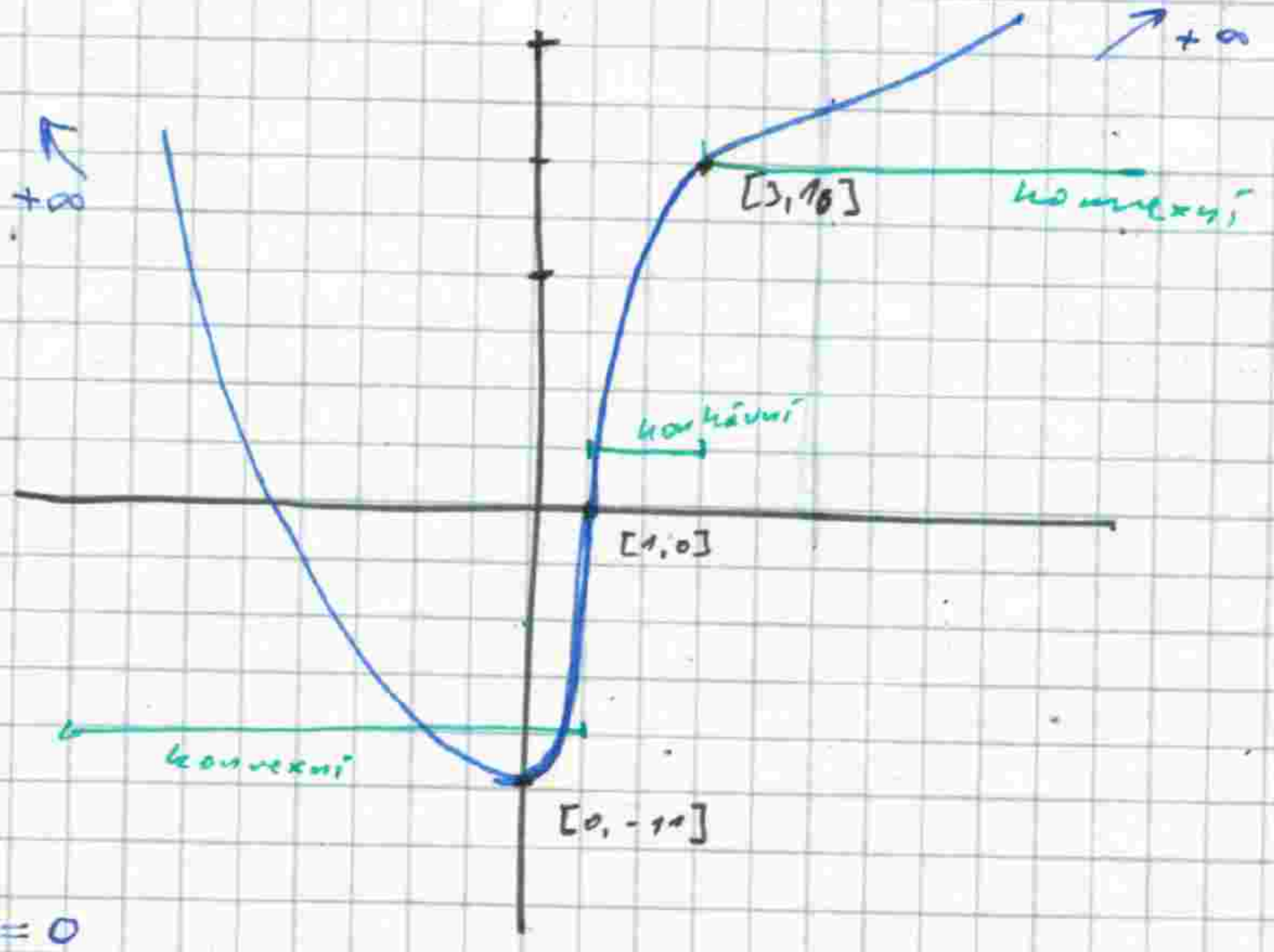
$x = 0: \quad F(0) = -11$

$\lim_{x \to \pm\infty} F(x) = +\infty$

Odhadneme kořeny. Pro $x = 1$ je $1 - 8 + 18 - 11 = 0$

$$x^4 - 8x^3 + 18x^2 - 11 : (x-1) = x^3 - 7x^2 + 11x + 11$$
$$\begin{array}{l} -x^4 + x^3 \\ \hline -7x^3 + 18x^2 - 11 \\ +7x^3 - 7x^2 \\ \hline 11x^2 - 11 \\ -11x^2 + 11x \\ \hline 11x - 11 \end{array}$$

$F(x) = (x-1)(x^3 - 7x^2 + 11x + 11)$



$F'(x) = 4x^3 - 24x^2 + 36x = 0$

$x = 0; \quad 4x^2 - 24x + 36 = 0 \qquad x^2 - 6x + 9 = 0$

$(x-3)^2 = 0 \quad \Rightarrow \quad x = 3$

$x = 10$: $4000 - 2400 + 360 = 1960 > 0$ +

−	0	+	3	+
klesá		roste		roste
	extrém		inflexní bod	

$F(3) = 3^4 - 8 \cdot 3^3 + 18 \cdot 3^2 - 11 = 81 - 8 \cdot 27 + 18 \cdot 9 - 11 = 81 - 216 + 162 - 11 = 16$

$F''(x) = 12x^2 - 48x + 36 = 0 \qquad x^2 - 4x + 3 = 0$

$$x = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2} = \begin{cases} 3 \\ 1 \end{cases}$$

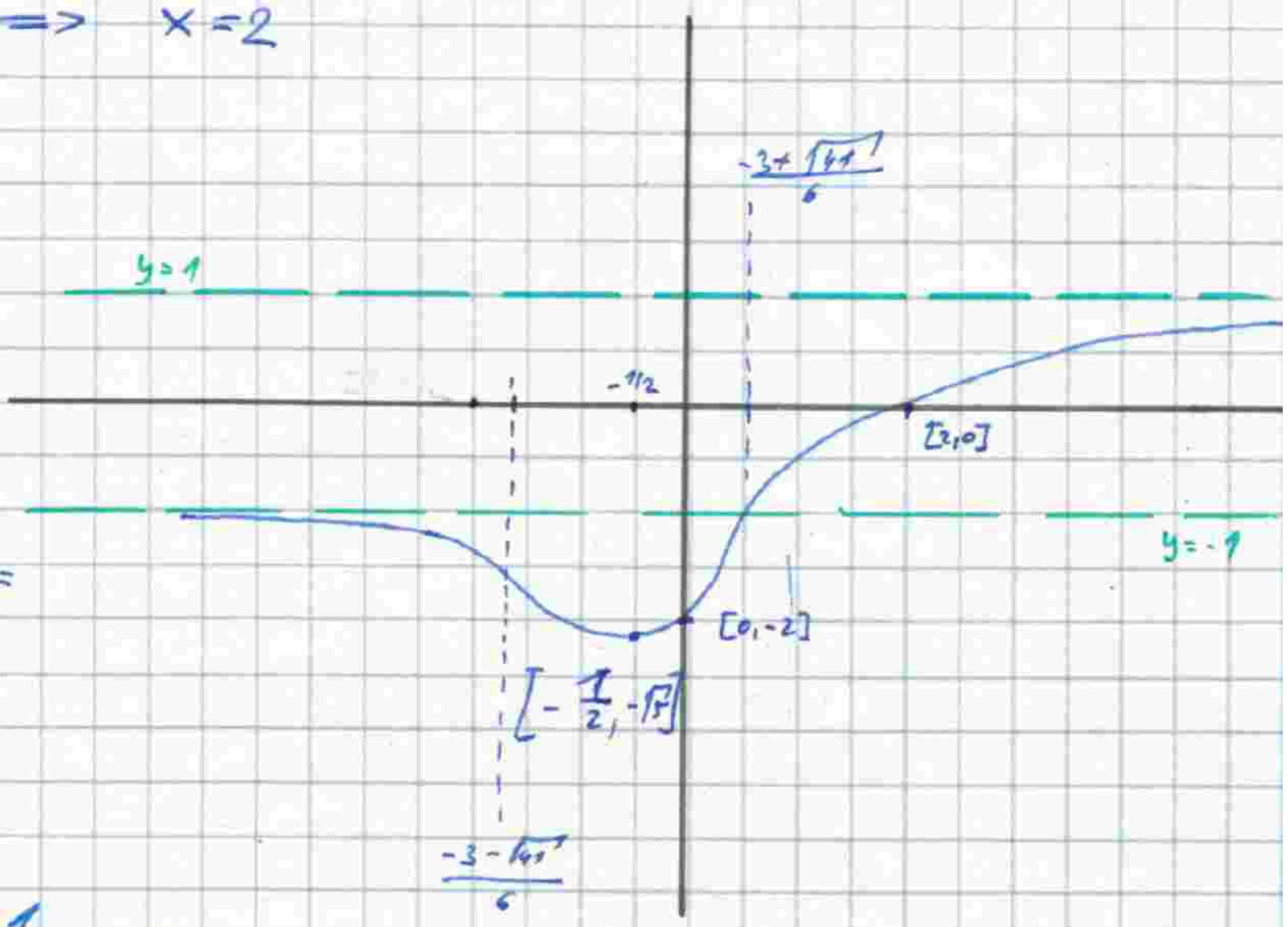
+	1	−	3	+
konvexní		konkávní		konvexní

• $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$ $D_f = \mathbb{R}$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x\left(1-\frac{2}{x}\right)}{|x|\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\pm\infty} \pm\frac{1-\overset{\to 0}{\frac{2}{x}}}{\sqrt{1+\underset{\to 0}{\frac{1}{x^2}}}} = \pm 1$$

Fce má vodorovnou asymptotu $y=1$ a $y=-1$.

$f(0) = -2$ $f(x) = 0 \iff x = 2$



$$f'(x) = \frac{\sqrt{x^2+1} - (x-2)\cdot\frac{2x}{2\sqrt{x^2+1}}}{x^2+1} =$$

$$= (x^2+1)^{-3/2}\cdot\left[x^2+1 - x(x-2)\right] =$$

$$= \frac{x^2 - x^2 + 2x + 1}{(x^2+1)^{3/2}} = \frac{2x+1}{(x^2+1)^{3/2}} = 0$$

$\iff x = -\frac{1}{2}$ $-$ | $+$ ($-1/2$): klesá, roste, min

$$f(-1/2) = \frac{-1/2 - 2}{\sqrt{1/4+1}} = -\frac{5/2}{\sqrt{5/4}} = -\frac{5\cdot 2}{\sqrt{5}\cdot 2} = -\sqrt{5} \doteq 2.2$$

$$f''(x) = \frac{2(x^2+1)^{3/2} - (2x+1)\cdot 2x\cdot\frac{3}{2}\cdot(x^2+1)^{1/2}}{(x^2+1)^3} = \frac{2(x^2+1) - (2x+1)\cdot 3x}{(x^2+1)^{5/2}} =$$

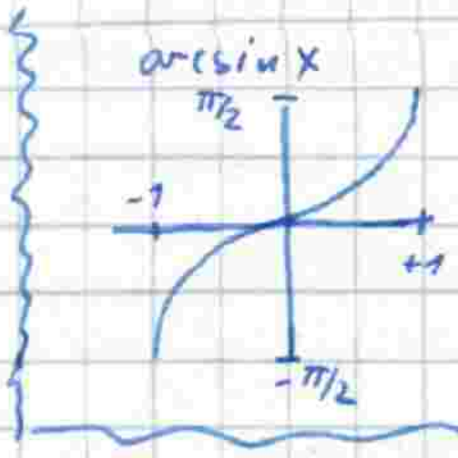
$$= \frac{2x^2+2-6x^2-3x}{(x^2+1)^{5/2}} = \frac{-4x^2-3x+2}{(x^2+1)^{5/2}} = 0$$

$4x^2+3x-2 = 0$ $x = \dfrac{-3 \pm \sqrt{9+32}}{6} = \dfrac{-3\pm\sqrt{41}}{6}$

Pozn.: Od ~~maxima~~ minima fce roste a k asymptotě $y=1$ se tedy nemůže přiblížit shora - to by musela klesat.

To samé v $y=-1$ (ovšem opačně).

• $F(x) = \arcsin \frac{2x}{x^2+1}$



$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$

$D_F: \; -1 \le \frac{2x}{x^2+1} \le +1$

a) $-1(x^2+1) \le 2x$
$0 \le x^2+2x+1$
pro $\forall x \in \mathbb{R}$

b) $2x \le x^2+1$
$0 \le x^2-2x+1$
pro $\forall x \in \mathbb{R}$

$\Rightarrow \; D_F = \mathbb{R}$

$F(0) = \arcsin 0 = 0$
$\Rightarrow$ graf prochází počátkem

$\lim_{x \to \pm\infty} F(x) = \lim_{y \to 0} \arcsin y = 0$

$\Rightarrow$ vodorovná asymptota $y = 0$

$$F'(x) = \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} \cdot \frac{1}{\sqrt{\left(\left(\frac{2x}{x^2+1}\right)^2 - 1\right) \cdot -1}} =$$

$$= \frac{-2x^2+2}{(x^2+1)^2} \cdot \frac{|x^2+1|}{\sqrt{4x^2-(x^2+1)^2}} = -2 \cdot \frac{x^2-1}{x^2+1} \cdot \frac{1}{(+x^4 - 4x^2 + 2x^2 + 1)^{1/2}} =$$

$$= -2 \, \frac{x^2-1}{x^2+1} \cdot (x^4 - 2x^2+1)^{1/2} =$$

$$= -2 \, \frac{x^2-1}{x^2+1} \cdot \frac{1}{\sqrt{(x^2-1)^2}} =$$

$$= -2 \cdot \frac{x^2-1}{x^2+1} \cdot \frac{1}{|x^2-1|}$$

Derivace v ± 1 neexistuje! Protože ale funkční hodnoty v těchto bodech existují:

$F(-1) = \arcsin(-1) = -\pi/2$
$F(+1) = \arcsin(+1) = +\pi/2$

−	-1	+	$+1$	−
klesá		roste		klesá

$F'(x) = 2k \cdot \frac{1}{x^2+1}$ $\quad k = \begin{cases} +1 & \text{pro } x \in (-1,+1) \\ -1 & \text{pro } x \in (-\infty,-1) \cup (+1,+\infty) \end{cases}$

Budou tam nějaké špičky. Existují i derivace zleva/zprava, což je další indikace špiček.

$$F''(x) = 2k \cdot 2x \cdot (-1) \cdot \frac{1}{(x^2+1)^2} = -\frac{4kx}{(x^2+1)^2} = 0 \text{ pro } x = 0$$

$k=-1$		$k=1$		$k=1$		$k=-1$
−	-1	+	0	−	$+1$	+
konkávní		konvexní		konkávní		konvexní